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Dreyer, Frédéric A ; Karlberg, Alexander

Abstract: We calculate the next-to-next-to-next-to-leading-order (N^3LO) QCD corrections to inclusive vector-boson fusion Higgs production at proton colliders, in the limit in which there is no color exchange between the hadronic systems associated with the two colliding protons. We also provide differential cross sections for the Higgs transverse momentum and rapidity distributions. We find that the corrections are at the 1‰-2‰ level, well within the scale uncertainty of the next-to-next-to-leading-order calculation. The associated scale uncertainty of the N^3LO calculation is typically found to be below the 2‰ level. We also consider theoretical uncertainties due to missing higher order parton distribution functions, and provide an estimate of their importance.

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Vector-boson fusion Higgs production at N^{3}LO in QCD

Frédéric A. Dreyer[1, 2, 3] and Alexander Karlberg[4]

[1] *Sorbonne Universités, UPMC Univ Paris 06, UMR 7589, LPTHE, F-75005, Paris, France*
[2] *CNRS, UMR 7589, LPTHE, F-75005, Paris, France*
[3] *CERN, Theoretical Physics Department, CH-1211 Geneva 23, Switzerland*
[4] *Rudolf Peierls Centre for Theoretical Physics, 1 Keble Road, University of Oxford, UK*

We calculate the next-to-next-to-next-to-leading-order (N^{3}LO) QCD corrections to inclusive vector-boson fusion (VBF) Higgs production at proton colliders, in the limit in which there is no colour exchange between the hadronic systems associated with the two colliding protons. We also provide differential cross sections for the Higgs transverse momentum and rapidity distributions. We find that the corrections are at the $1-2$‰ level, well within the scale-uncertainty of the next-to-next-to-leading-order (NNLO) calculation. The associated scale-uncertainty of the N^{3}LO calculation is typically found to be below the 2‰ level. We also consider theoretical uncertainties due to missing higher order parton distribution functions, and provide an estimate of their importance.

Since the discovery of the Higgs boson [1, 2], the LHC has commenced a program of precision studies of its properties. Higgs production through vector-boson fusion (VBF), shown in figure 1, is a key process for precision measurements of properties of the Higgs boson [3], as it is a clean channel with very distinctive kinematics, due to its t-channel production and to the presence of two high rapidity jets in the final state. These features provide an ideal access for the intricate measurements of the Higgs couplings [4]. Currently the VBF production signal strength has been measured with a precision of about 24% [5], though significant improvements can be expected during run 2 and with the high luminosity LHC.

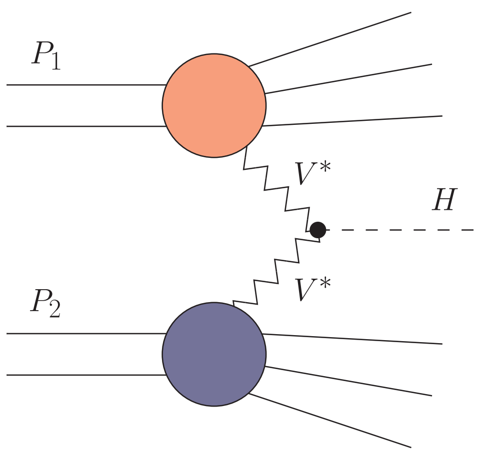

FIG. 1. Illustration of Higgs production through vector-boson fusion.

In order to experimentally determine the properties of the Higgs boson it is crucial to have very precise theoretical predictions for cross sections. The inclusive cross section for VBF Higgs production is known to next-to-next-to-leading order (NNLO) [6, 7] in the structure function approach, in which VBF-induced Higgs production is treated as a double deep-inelastic scattering (DIS) process [8]. This calculation found NNLO corrections of about 1% and renormalisation and factorisation scale uncertainties at the 5‰ level. Recently, the fully differential NNLO QCD corrections in VBF Higgs production have been computed [9]. These where found to be significant after typical VBF cuts, with corrections up to 10–12% in certain kinematical regions. The calculation also showed no significant reduction in the associated scale uncertainties compared to the scale uncertainty at next-to-leading-order (NLO).

The structure function approximation is known to be very accurate for VBF, because non-factorisable colour exchanges are both kinematically and colour suppressed, such that they are expected to contribute to less than 1% of the cross section [7, 10, 11]. This approach is exact in the limit in which one considers that there are two identical copies of QCD associated with each of the two protons (shown orange and blue in figure 1), whose interaction is mediated by the weak force.

In this letter we compute the next-to-next-to-next-to-leading order (N^{3}LO) QCD corrections to the inclusive cross section in the structure function approach. This calculation provides the second N^{3}LO calculation for processes of relevance to the LHC physics program, after a similar accuracy was recently achieved in the gluon-gluon fusion channel [12]. It represents an important milestone towards achieving a fully differential N^{3}LO calculation with the projection-to-Born method [9]. We also provide an estimate of contributions to the cross section from missing higher order parton distribution functions (PDFs) as these are currently only known at NNLO.

In the structure function approach the VBF Higgs production cross section is calculated as a double DIS process and can thus be expressed as [8]

$$
\begin{aligned}
d\sigma =& \frac{4\sqrt{2}}{s} G_F^3 m_V^8 \Delta_V^2(Q_1^2)\Delta_V^2(Q_2^2) \\
&\times \mathcal{W}_{\mu\nu}^V(x_1, Q_1^2)\mathcal{W}^{V,\mu\nu}(x_2, Q_2^2) d\Omega_{\rm VBF}\,. \qquad (1)
\end{aligned}
$$

Here G_F is Fermi's constant, m_V is the mass of the vector boson, $\sqrt{s}$ is the collider centre-of-mass energy, Δ_V^2 is the squared boson propagator, $Q_i^2 = -q_i^2$ and $x_i = Q_i^2/(2P_i \cdot q_i)$ are the usual DIS variables, and $d\Omega_{\rm VBF}$ is the three-particle VBF phase space. The hadronic tensor $\mathcal{W}^V_{\mu\nu}$ can be expressed as

$$\mathcal{W}^V_{\mu\nu}(x_i, Q_i^2) = \Big(- g_{\mu\nu} + \frac{q_{i,\mu}q_{i,\nu}}{q_i^2}\Big) F_1^V(x_i, Q_i^2) \\ + \frac{\hat{P}_{i,\mu}\hat{P}_{i,\nu}}{P_i \cdot q_i} F_2^V(x_i, Q_i^2) + i\epsilon_{\mu\nu\rho\sigma}\frac{P_i^\rho q_i^\sigma}{2P_i \cdot q_i} F_3^V(x_i, Q_i^2)\,, \quad (2)$$

where we defined $\hat{P}_{i,\mu} = P_{i,\mu} - \frac{P_i \cdot q_i}{q_i^2} q_{i,\mu}$, and the $F_i^V(x, Q^2)$ functions are the standard DIS structure functions with $i = 1, 2, 3$ and $V = Z, W^-, W^+$.

From the knowledge of the vector-boson momenta q_i, it is straightforward to reconstruct the Higgs momentum. As such, the cross section obtained using equation (1) is differential in the Higgs kinematics.

In order to compute the N^{n}LO cross section, we require the structure functions F_i^V up to order $\mathcal{O}(\alpha_s^n)$ in the strong coupling constant. We express the structure functions as convolutions of the PDFs with the short distance coefficient functions

$$F_i^V = \sum_{a=q,g} C_i^{V,a} \otimes f_a\,, \quad i = 1, 2, 3\,. \quad (3)$$

All the necessary coefficient functions are known up to third order.[1] To compute the N^{3}LO VBF Higgs production cross section, one can therefore evaluate the convolution of the PDF with the appropriate coefficient functions in equation (3). At N^{3}LO, additional care is required due to the appearance of new flavour topologies [14]. As such, contributions corresponding to interferences of diagrams where the vector boson attaches on different quark lines are to be set to zero for charged boson exchanges.

To compute the dependence of the cross section on the values of the factorisation and renormalisation scales, we use renormalisation group methods [15–17], and evaluate the scale dependence to third order in the coefficient functions as well as in the PDFs. The running of the coefficient functions can be obtained using the first two terms in the expansion of the beta function. To obtain the dependence of the PDFs on the factorisation scale, we integrate the parton density evolution equation. For completeness, the technical details of this procedure are given in the supplemental material of this letter [18].

There is one source of formally N^{3}LO QCD corrections appearing in equation (3) which is currently unknown, namely missing higher order terms in the determination of the PDF. Indeed, while one would ideally calculate the N^{3}LO cross section using N^{3}LO parton densities, only NNLO PDF sets are available at this time. These will be missing contributions from two main sources: from the higher order corrections to the coefficient functions that relate physical observables to PDFs; and from the higher order splitting functions in the evolution of the PDFs.

To evaluate the impact of future N^{3}LO PDF sets on the total cross section, we consider two different approaches. A first, more conservative estimate, is to derive the uncertainty related to higher order PDF sets from the difference at lower orders, as described in [19] (see also [20]). We compute the NNLO cross section using both the NLO and the NNLO PDF set, and use their difference to extract the N^{3}LO PDF uncertainty. We find in this way that at 13 TeV the uncertainty from missing higher orders in the extractions of PDFs is

$$\delta_A^{\mathrm{PDF}} = \frac{1}{2}\left|\frac{\sigma^{\mathrm{NNLO}}_{\text{NNLO-PDF}} - \sigma^{\mathrm{NNLO}}_{\text{NLO-PDF}}}{\sigma^{\mathrm{NNLO}}_{\text{NNLO-PDF}}}\right| = 1.1\%\,. \quad (4)$$

Because the convergence is greatly improved going from NNLO to N^{3}LO compared to one order lower, one might expect this to be rather conservative even with the factor half in equation (4). Therefore, we also provide an alternative estimate of the impact of higher orders PDFs, using the known N^{3}LO F_2 structure function.

We start by rescaling all the parton distributions using the F_2 structure function evaluated at a low scale Q_0.

$$f^{\mathrm{N^3LO,approx.}}(x, Q) = f^{\mathrm{NNLO}}(x, Q)\frac{F_2^{\mathrm{NNLO}}(x, Q_0)}{F_2^{\mathrm{N^3LO}}(x, Q_0)}\,. \quad (5)$$

In practice, we will use the Z structure function. We then re-evaluate the structure functions in equation (3) using the approximate higher order PDF given by equation (5). This yields

$$\delta_B^{\mathrm{PDF}}(Q_0) = \left|\frac{\sigma^{\mathrm{N^3LO}} - \sigma^{\mathrm{N^3LO}}_{\text{rescaled}}(Q_0)}{\sigma^{\mathrm{N^3LO}}}\right| = 7.9‰\,, \quad (6)$$

where in the last step, we used $Q_0 = 8$ GeV and considered 13 TeV proton collisions.

By calculating a rescaled NLO PDF and evaluating the NNLO cross section in this way, we can evaluate the ability of this method to predict the corrections from NNLO PDFs. We find that with $Q_0 = 8$ GeV, the uncertainty estimate obtained in this way captures relatively well the impact of NNLO PDF sets.

The rescaled PDF sets obtained using equation (5) will be missing N^{3}LO corrections from the evolution of the PDFs in energy. We have checked the impact of these terms by varying the renormalisation scale up and down by a factor two around the factorisation scale in the splitting functions used for the PDF evolution. We find that the theoretical uncertainty associated with missing higher order splitting functions is less than one permille of the total cross section. Comparing this with equation (6), it is clear that these effects are numerically subleading, suggesting that a practical alternative to full N^{3}LO PDF sets could be obtained by carrying out a fit of DIS data using the hard N^{3}LO matrix element. We leave a detailed study of this question for future work.

[1] The even-odd differences between charged-current coefficient functions are known only approximately, since only the five lowest moments have been calculated [13]. However, the uncertainty associated with this approximation is less than 1‰ of the N^{3}LO correction, and therefore completely negligible.

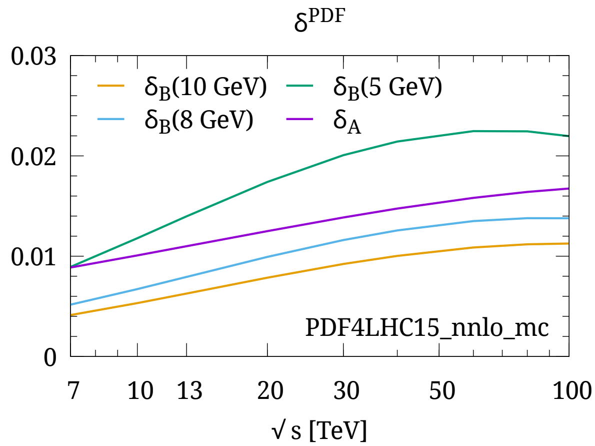

FIG. 2. Estimate of the impact of missing higher orders corrections in PDFs, using equations (4) and (6) with $Q_0 = 5$, 8 and 10 GeV.

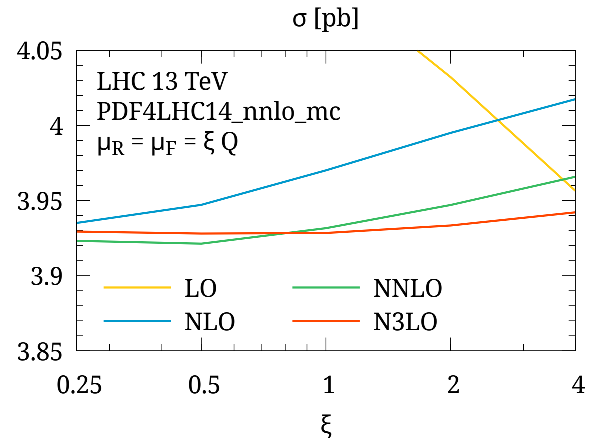

FIG. 3. Dependence of the cross section on the renormalisation and factorisation scales for each order in perturbation theory.

The uncertainty estimates obtained with the two different methods described by equations (4) and (6) is shown in figure 2 as a function of center-of-mass energy, and for a range of Q_0 values.

One should note that the uncertainty estimates given in equations (4) and (6) do not include what is usually referred to as PDF uncertainties. While we are here calculating missing higher order uncertainties to NNLO PDF sets, typical PDF uncertainties correspond to uncertainties due to errors on the experimental data and limitations of the fitting procedure. These can be evaluated for example with the PDF4LHC15 prescription [21], and are of about 2% at 13 TeV, which is larger than the corrections discussed above. One can also combine them with α_s uncertainties, which are at the 5‰ level.

Let us now discuss in more detail phenomenological consequences of the N^{3}LO corrections to VBF Higgs production. We present results for a wide range of energies in proton-proton collisions. The central factorisation and renormalisation scales are set to the squared momentum of the corresponding vector boson. To estimate missing higher-order uncertainties, we use a seven-point scale variation, varying the scales by a factor two up and down while keeping $0.5 < \mu_R/\mu_F < 2$

$$\mu_{R,i} = \xi_{\mu_R} Q_i \,, \quad \mu_{F,i} = \xi_{\mu_F} Q_i \,, \tag{7}$$

where $\xi_{\mu_R}, \xi_{\mu_F} \in \left\{\frac{1}{2}, 1, 2\right\}$ and $i = 1$, 2 corresponds to the upper and lower hadronic sectors.

Our implementation of the calculation is based on the inclusive part of `proVBFH` which was originally developed for the differential NNLO VBF calculation [9]. We have used the phase space from `POWHEG`'s two-jet VBF Higgs calculation [22]. The matrix element is derived from structure functions obtained with the parametrised DIS coefficient functions [13, 14, 16, 23–29], evaluated using `HOPPET` v1.2.0-devel [30].

For our computational setup, we use a diagonal CKM matrix with five light flavours ignoring top-quarks in the internal lines and final states. Full Breit-Wigner propagators for the W, Z and the narrow-width approximation for the Higgs boson are applied. We use the PDF4LHC15_nnlo_mc PDF [21, 31–33] and four-loop evolution of the strong coupling, taking as our initial condition $\alpha_s(M_Z) = 0.118$. We set the Higgs mass to $M_H = 125.09$ GeV, in accordance with the experimentally measured value [34]. Electroweak parameters are obtained from their PDG [35] values and tree-level electroweak relations. As inputs we use $M_W = 80.385$ GeV, $M_Z = 91.1876$ GeV and $G_F = 1.16637 \times 10^{-5}$ GeV^{-2}. For the widths of the vector bosons we use $\Gamma_W = 2.085$ GeV and $\Gamma_Z = 2.4952$ GeV.

To study the convergence of the perturbative series, we show in figure 3 the inclusive cross section obtained at 13 TeV with $\mu_R = \mu_F = \xi Q$ for $\xi \in [1/4, 4]$. Here we observe that at N^{3}LO the scale dependence becomes extremely flat over the full range of renormalisation and factorisation scales. We note that similarly to the results obtained in the the gluon-fusion channel [12], the convergence improves significantly at N^{3}LO, with the N^{3}LO prediction being well inside of the NNLO uncertainty band, while at lower orders there is a pattern of limited overlap of theoretical uncertainties.

In figure 4 (left), we give the cross section as a function of center-of-mass energy. We see that at N^{3}LO the convergence of the perturbative series is very stable, with corrections of about 1‰ on the NNLO result. The scale uncertainty is dramatically reduced, going at 13 TeV from 7‰ at NNLO to 1.4‰ at N^{3}LO. A detailed breakdown of the cross section and scale uncertainty obtained at each order in QCD is given in table I for $\sqrt{s} = 13$, 14 and 100 TeV.

The center and right plots of figure 4 show the Higgs transverse momentum and rapidity distributions at each

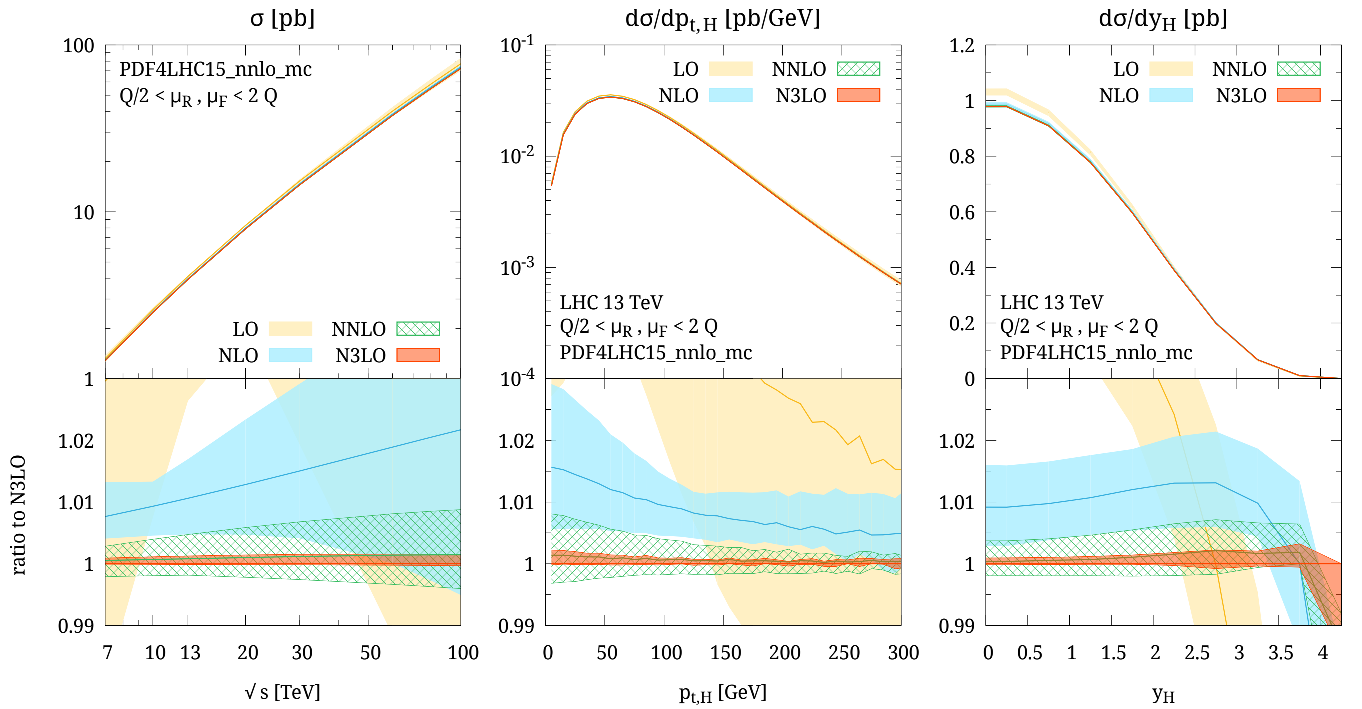

FIG. 4. Cross section as a function of center-of-mass energy (left), Higgs transverse momentum distribution (center) and Higgs rapidity distribution (right).

	$\sigma^{(13\text{ TeV})}$ [pb]	$\sigma^{(14\text{ TeV})}$ [pb]	$\sigma^{(100\text{ TeV})}$ [pb]
LO	$4.099\,^{+0.051}_{-0.067}$	$4.647\,^{+0.037}_{-0.058}$	$77.17\,^{+6.45}_{-7.29}$
NLO	$3.970\,^{+0.025}_{-0.023}$	$4.497\,^{+0.032}_{-0.027}$	$73.90\,^{+1.73}_{-1.94}$
NNLO	$3.932\,^{+0.015}_{-0.010}$	$4.452\,^{+0.018}_{-0.012}$	$72.44\,^{+0.53}_{-0.40}$
$\mathrm{N^3LO}$	$3.928\,^{+0.005}_{-0.001}$	$4.448\,^{+0.006}_{-0.001}$	$72.34\,^{+0.11}_{-0.02}$

TABLE I. Inclusive cross sections at LO, NLO, NNLO and $\mathrm{N^3LO}$ for VBF Higgs production. The quoted uncertainties correspond to scale variations $Q/2 < \mu_R, \mu_F < 2Q$, while statistical uncertainties are at the level of 0.2‰.

order in QCD, where we observe again a large reduction of the theoretical uncertainty at $\mathrm{N^3LO}$.

A comment is due on non-factorisable QCD corrections. Indeed, for the results presented in this letter, we have considered VBF in the usual DIS picture, ignoring diagrams that are not of the type shown in figure 1. These effects neglected by the structure function approximation are known to contribute less than 1% to the total cross section at NNLO [7]. The effects and their relative corrections are as follows:

- Gluon exchanges between the upper and lower hadronic sectors, which appear at NNLO, but are kinematically and colour suppressed. These contributions along with the heavy-quark loop induced contributions have been estimated to contribute at the permille level [7].
- t-/u-channel interferences which are known to contribute $\mathcal{O}(5‰)$ at the fully inclusive level and $\mathcal{O}(0.5‰)$ after VBF cuts have been applied [10].
- Contributions from s-channel production, which have been calculated up to NLO [10]. At the inclusive level these contributions are sizeable but they are reduced to $\mathcal{O}(5‰)$ after VBF cuts.
- Single-quark line contributions, which contribute to the VBF cross section at NNLO. At the fully inclusive level these amount to corrections of $\mathcal{O}(1\%)$ but are reduced to the permille level after VBF cuts have been applied [11].
- Loop induced interferences between VBF and gluon-fusion Higgs production. These contributions have been shown to be much below the permille level [36].

Furthermore, for phenomenological applications, one also needs to consider NLO electroweak effects [10], which amount to $\mathcal{O}(5\%)$ of the total cross section. We leave a detailed study of non-factorisable and electroweak effects for future work. The code used for this calculation will be published in the near future [37].

In this letter, we have presented the first $\mathrm{N^3LO}$ calculation of a $2 \to 3$ hadron-collider process, made possible by the DIS-like factorisation of the process. This brings the precision of VBF Higgs production to the same formal accuracy as was recently achieved in the gluon-gluon fusion channel in the heavy top mass approximation [12]. The

N^3LO corrections are found to be tiny, $1-2‰$, and well within previous theoretical uncertainties, but they provide a large reduction of scale uncertainties, by a factor 5. This calculation also provides the first element towards a differential N^3LO calculation for VBF Higgs production, which could be achieved through the projection-to-Born method [9] using a NNLO DIS 2+1 jet calculation [38].

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SUPPLEMENTAL MATERIAL

SUPPLEMENTAL MATERIAL TO "VECTOR-BOSON FUSION HIGGS PRODUCTION AT N^{3}LO IN QCD" BY FRÉDÉRIC A. DREYER AND ALEXANDER KARLBERG

1. Scale variation up to N^{3}LO

To compute the dependence of the cross section on the values of the factorisation and renormalisation scales, we use renormalisation group methods [15, 16] on the structure functions

$$F_i^V = \sum_a C_i^{V,a} \otimes f_a\,, \quad i = 1, 2, 3\,. \tag{8}$$

This requires us to compute the scale dependence to third order in the coefficient functions as well as in the PDFs.

We start by evaluating the running coupling for α_s

$$\alpha_s(Q) = \alpha_s(\mu_R) + \alpha_s^2(\mu_R)\beta_0 L_{RQ} + \alpha_s^3(\mu_R)(\beta_0^2 L_{RQ}^2 + \beta_1 L_{RQ}) + \mathcal{O}(\alpha_s^4(\mu_R))\,, \tag{9}$$

where we introduced the shorthand notation

$$L_{RQ} = \ln\left(\frac{\mu_R^2}{Q^2}\right)\,, \quad L_{FQ} = \ln\left(\frac{\mu_F^2}{Q^2}\right)\,, \tag{10}$$

as well as $\beta_0 = (33 - 2n_f)/12\pi$ and $\beta_1 = (153 - 19n_f)/24\pi^2$. The coefficient functions can thus be expressed as an expansion in $\alpha_s(\mu_R)$

$$\begin{aligned} C_i^{(0)} + \frac{\alpha_s(Q)}{2\pi} C_i^{(1)} + \left(\frac{\alpha_s(Q)}{2\pi}\right)^2 C_i^{(2)} + \left(\frac{\alpha_s(Q)}{2\pi}\right)^3 C_i^{(3)} &= C_i^{(0)} + \frac{\alpha_s(\mu_R)}{2\pi} C_i^{(1)} + \left(\frac{\alpha_s(\mu_R)}{2\pi}\right)^2 \left(C_i^{(2)} + 2\pi\beta_0 C_i^{(1)} L_{RQ}\right) \\ &+ \left(\frac{\alpha_s(\mu_R)}{2\pi}\right)^3 \left[C_i^{(3)} + 4\pi\beta_0 C_i^{(2)} L_{RQ} + 4\pi^2 C_i^{(1)} L_{RQ}(\beta_1 + \beta_0^2 L_{RQ})\right]. \end{aligned} \tag{11}$$

To evaluate the dependence of the PDFs on the factorisation scale μ_F, we integrate the DGLAP equation, using

$$f(x, Q) = f(x, \mu_F) - \int_0^{L_{FQ}} dL \frac{d}{dL} f(x, \mu)\,. \tag{12}$$

It is then straightforward to express the PDF in terms of an expansion in $\alpha_s(\mu_R)$ evaluated at μ_F. We have

$$\begin{aligned} f(x,Q) = f(x,\mu_F) &- \frac{\alpha_s(\mu_R)}{2\pi} L_{FQ} f(x,\mu_F) P^{(0)} - \left(\frac{\alpha_s(\mu_R)}{2\pi}\right)^2 L_{FQ} f(x,\mu_F) \left[P^{(1)} - \frac{1}{2} L_{FQ} (P^{(0)})^2 - \pi\beta_0 P^{(0)} (L_{FQ} - 2L_{RQ})\right] \\ &- \left(\frac{\alpha_s(\mu_R)}{2\pi}\right)^3 L_{FQ} f(x,\mu_F) \Big[P^{(2)} - \frac{1}{2} L_{FQ} (P^{(0)} P^{(1)} + P^{(1)} P^{(0)}) + \pi\beta_0 (L_{FQ} - 2L_{RQ})(L_{FQ} (P^{(0)})^2 - 2P^{(1)}) \\ &+ \frac{1}{6} L_{FQ}^2 (P^{(0)})^3 + 4\pi^2 \beta_0^2 P^{(0)} (L_{RQ}^2 - L_{FQ} L_{RQ} + \frac{1}{3} L_{FQ}^2) - 2\pi^2 \beta_1 P^{(0)} (L_{FQ} - 2L_{RQ})\Big]\,. \end{aligned} \tag{13}$$

Here we defined the expansion of the splitting functions as

$$P(z, \alpha_s) = \sum_{i=0} \left(\frac{\alpha_s}{2\pi}\right)^i P^{(i)}(z)\,, \tag{14}$$

where terms up to $P^{(2)}(z)$ are known [39, 40]. Equations (11) and (13) allow us to evaluate the convolution in equation (8) up to N^{3}LO in perturbative QCD for any choice of the renormalisation and factorisation scales.